ON THE DEVELOPMENT OF THE FLOW OF A VISCOUS HEAT-CONDUCTING GAS IN A PIPE

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In the monograph of Targ [1] the flow of a viscous incompressible fluid is treated in terms of approximate differential equations which take the viscous and transport terms partially into account, and solutions are given for a number of problems. The solution for the development of the flow in a circular cylindrical pipe shows a satisfactory agreement with the results of experiments, not only with respect to the length of the entrance region, but also with respect to the development of the velocity profile at stations in the entrance region. The same approximate equations were used by Ovchinnikov [2] to solve the problem of the development of the flow in a diffuser for an arbitrary velocity distribution at the entrance. In this case the calculated results were well verified qualitatively by special experiments carried out by the author, even for Reynolds numbers as large as 25×10^4.

The cases just described suggest that suitable approximate equations might be set up to take the nonlinear terms partially into account, and that these equations might be used for the solution of various problems in the flow of a viscous heat-conducting gas. In the literature such problems have so far been treated by a method used by Schiller [3] to study the flow of an incompressible fluid; i.e. by joining the constant-velocity profile in the core with the velocity profile determined separately for the boundary layer (for example, in the article of Kaul and Brown [4]). Solutions for these problems on the basis of approximate linearized equations have not yet been presented in the literature, and such solutions may have theoretical and practical interest in the design of diffusers for engines on moving objects and in the calculation of certain losses associated with gas flow in ducts and diffusers.

1. We will assume two-dimensional stationary motion of an ideal gas having the Clapeyron equation as equation of state and having constant

specific heats. By analogy with another paper [5], we will introduce dimensionless variables and parameters in the following manner;

$$x = lx_1, \quad y = \varepsilon l y_1, \quad u = U_0 u_1, \quad v = \varepsilon U_0 v_1, \quad p = p_0 p_1$$

$$\rho = \rho_0\rho_1, \quad T = T_0 T_1, \quad \mu = \mu_0 \mu_1, \quad \varkappa = \varkappa_0 \varkappa_1 \tag{1.1}$$

$$M^2 = \frac{U_0^2 \rho_0}{\gamma p_0}, \quad R = \frac{\rho_0 U_0 l}{\mu_0}, \quad P = \frac{\mu_0 g C_p}{\varkappa_0}$$

$$\gamma = \frac{C_p}{C_v}, \quad A\frac{U_0^2}{C_p g T_0} = (\gamma - 1) M^2$$

Using (1.1) and neglecting body forces, the differential equations for a viscous heat-conducting gas take the form

$$\begin{aligned}
&R\varepsilon^2\rho_1\left(u_1\frac{\partial u_1}{\partial x_1} + v_1\frac{\partial u_1}{\partial y_1}\right) = -\frac{R\varepsilon^2}{\gamma M^2}\frac{\partial p_1}{\partial x_1} + \frac{\partial}{\partial y_1}\left(\mu_1\frac{\partial u_1}{\partial y_1}\right) + \\
&\qquad + \varepsilon^2\left\{\frac{\partial}{\partial x_1}\left[\mu_1\left(\frac{4}{3}\frac{\partial u_1}{\partial x_1} - \frac{2}{3}\frac{\partial u_1}{\partial y_1}\right)\right] + \frac{\partial}{\partial y_1}\left(\mu_1\frac{\partial v_1}{\partial x_1}\right)\right\} \\
&\varepsilon^2\rho_1\left(u_1\frac{\partial v_1}{\partial x_1} + v_1\frac{\partial v_1}{\partial y_1}\right) = -\frac{1}{\gamma M^2}\frac{\partial p_1}{\partial y_1} + \frac{1}{R}\frac{\partial}{\partial x_1}\left(\mu_1\frac{\partial u_1}{\partial y_1}\right) + \\
&\qquad + \frac{\varepsilon^2}{R}\frac{\partial}{\partial x_1}\left(\mu_1\frac{\partial v_1}{\partial x_1}\right) + \frac{1}{R}\frac{\partial}{\partial y_1}\left[\mu_1\left(\frac{4}{3}\frac{\partial v_1}{\partial y_1} - \frac{2}{3}\frac{\partial u_1}{\partial x_1}\right)\right] \\
&R\varepsilon^2\rho_1\left(u_1\frac{\partial T_1}{\partial x_1} + v_1\frac{\partial T_1}{\partial y_1}\right) = \varepsilon^2 R\frac{\gamma-1}{\gamma}\left(u_1\frac{\partial p_1}{\partial x_1} + v_1\frac{\partial p_1}{\partial y_1}\right) + \frac{1}{P}\frac{\partial}{\partial y_1}\left(\varkappa_1\frac{\partial T_1}{\partial y_1}\right) + \\
&\qquad + \frac{\gamma-1}{\gamma}\mu_1\left(\frac{\partial u_1}{\partial y_1}\right)^2 + \frac{\varepsilon^2}{P}\frac{\partial}{\partial x_1}\left(\varkappa_1\frac{\partial T_1}{\partial x_1}\right) + \frac{\gamma-1}{\gamma}\mu_1\varepsilon^2\left\{\varepsilon^2\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \right. \\
&\qquad \left. + \frac{4}{3}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial v_1}{\partial y_1}\right)^2 + \frac{3}{2}\frac{\partial v_1}{\partial x_1}\frac{\partial u_1}{\partial y_1} - \frac{1}{2}\frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial y_1}\right]\right\} \quad \frac{\partial(\rho_1 v_1)}{\partial x_1} + \frac{\partial(\rho_1 v_1)}{\partial y_1} = 0
\end{aligned} \tag{1.2}$$

$$p_1 = \rho_1 T_1, \quad \mu_1 = \mu_1(T_1), \quad \varkappa_1 = \varkappa_1(T_1)$$

These differential equations (1.2) are identically satisfied by a strictly parallel flow with constant velocity U_0 and with constant-state variables, and for this case we may put

$$u_1 = 1, \quad v_1 = 0, \quad p_1 = 1, \quad \rho_1 = 1, \quad T_1 = 1, \quad \mu_1 = 1, \quad \varkappa_1 = 1 \tag{1.3}$$

We will now restrict ourselves to cases in which the presence of solid walls exerts a relatively small disturbing influence on the departure of all the variables of the gas flow from their values in the undisturbed stream. If ϵ is a small parameter, and if the dimensionless variables R, M, and P are assumed at the outset to vary like powers of ϵ, then the solution of equations (1.3) may be developed as infinite series in powers of the small parameter ϵ. We will assume that in certain special cases we may terminate these series with terms containing the first power of the small parameter; that is,

$$u_1 = 1 + \varepsilon u', \quad v_1 = \varepsilon v', \quad \rho_1 = 1 + \varepsilon\rho', \quad p_1 = 1 + \varepsilon p'$$
$$T_1 = 1 + \varepsilon T', \quad \mu_1 = 1 + \varepsilon\mu', \quad \varkappa_1 = 1 + \varepsilon\varkappa' \qquad (1.4)$$

We will further assume that the characteristic parameters have the following orders of magnitude;

$$R \sim \frac{1}{\varepsilon^2}, \qquad M \sim 1, \qquad P \sim 1 \qquad (1.5)$$

Substituting equations (1.4) into (1.2), taking account of (1.5), and equating the coefficients of terms linear in the small parameter ϵ, we obtain the following equations;

$$\frac{\partial u'}{\partial x_1} = -\frac{1}{\gamma M^2}\frac{\partial p'}{\partial x_1} + \frac{1}{\varepsilon^2 R}\frac{\partial^2 u'}{\partial y_1^2}, \qquad 0 = \frac{\partial p'}{\partial y_1}$$
$$\frac{\partial T'}{\partial x_1} = \frac{\gamma - 1}{\gamma}\frac{\partial p'}{\partial x_1} + \frac{1}{PR\varepsilon^2}\frac{\partial^2 T'}{\partial y_1^2} \qquad (1.6)$$
$$\frac{\partial \rho'}{\partial x_1} + \frac{\partial u'}{\partial x_1} + \frac{\partial v'}{\partial y_1} = 0, \qquad p' = \rho' + T'$$

Using the equation of state (1.6), the density ρ may be eliminated from the continuity equation, which finally takes the form

$$\frac{\partial p'}{\partial x_1} - \frac{\partial T'}{\partial x_1} + \frac{\partial u'}{\partial x_1} + \frac{\partial v'}{\partial y_1} = 0 \qquad (1.7)$$

Returning in equations (1.6) and (1.7) to the original dimensional variables; that is, putting

$$x_1 = \frac{x}{l}, \quad y_1 = \frac{y}{\varepsilon l}, \quad u' = \frac{u - U_0}{\varepsilon U_0}, \quad v' = \frac{v}{\varepsilon^2 U_0}$$
$$p' = \frac{p - p_0}{\varepsilon p_0}, \quad T' = \frac{T - T_0}{\varepsilon T_0}, \quad p_0 = g\rho_0 C_p \frac{\gamma - 1}{\gamma} T_0 \qquad (1.8)$$

we obtain the following approximate linearized equations of motion for a viscous heat-conducting gas;

$$gC_p\rho_0 U_0 \frac{\partial T}{\partial x} = AU_0 \frac{\partial p}{\partial x} + \varkappa_0 \frac{\partial^2 T}{\partial y^2}, \quad \rho_0 U_0 \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \mu_0 \frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + U_0\left(\frac{1}{p_0}\frac{\partial p}{\partial x} - \frac{1}{T_0}\frac{\partial T}{\partial x}\right) = 0, \qquad \frac{\partial p}{\partial y} = 0 \qquad (1.9)$$

If we compare the approximate equations (1.9) with the familiar equations for the plane boundary layer in a viscous compressible gas ([6], p. 282), we find that in our equations the transport terms for u, p, and T are partially accounted for in the manner of Oseen; moreover, the coefficients μ and κ are taken as constants, and the terms for the dissipation of energy are discarded. Equations (1.9) are therefore rather rough approximate equations, and their solutions will not be as accurate for phenomena in pipe flow or in thin boundary leyrs as the solutions of the corresponding boundary-layer equations. But a similar conclusion concerning the roughness of the approximation to the original equations applies for the equations employed in the solution of certain problems of the three-dimensional boundary layer for the flow of an incompressible fluid. Nevertheless, the solutions obtained in these problems, as was shown above, are not in bad agreement with the results of experiment or with the solutions of more exact equations. Therefore we may assume that for a number of problems of gas flow in thin layers or in pipes the solution of equations (1.9) will give a correct picture of the flow, not only in a qualitative but also in a quantitative way.

For problems concerning the boundary layer near a body, the pressure p entering into (1.9) may be considered a given function of the coordinate x measured along a contour of the profile. For problems concerning the development of the flow in pipes, the boundary conditions for the lateral velocity v allow a differential equation for the pressure to be obtained.

2. If we assume that the motion of the gas is axially symmetric and if we take the transversal velocity component as zero, then the approximate equations analogous to (1.9) will have the following form;

$$\begin{gathered}\rho_0 U_0 \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \frac{\mu_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \qquad \frac{\partial p}{\partial r} = 0 \\ \frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + U_0\left(\frac{1}{p_0}\frac{\partial p}{\partial x} - \frac{1}{T_0}\frac{\partial T}{\partial x}\right) = 0 \\ g C_p \rho_0 U_0 \frac{\partial T}{\partial x} = A U_0 \frac{\partial p}{\partial x} + \frac{\varkappa_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\end{gathered} \tag{2.1}$$

We will apply these approximate equations (2.1) to a special case, namely to a circular cylindrical pipe with an open leading edge. Let the pipe move in a fluid with velocity U_0 parallel to its axis of symmetry. Let the pressure inside the pipe at a certain distance from the entrance be maintained by some means at a value p_2 smaller than the pressure p_0 in the fluid far ahead of the pipe entrance. Because of the evacuating effect of the pressure difference, a certain flow will arise in the fluid ahead of the leading edge, as well as a relative motion for the fluid inside the pipe. If we interchange the motion of the fluid and the pipe, then the flow at an infinite distance from the leading edge of the stationary pipe will have a velocity U_0 in the direction of the positive x-axis; the pressure will be p_0, and the temperature will be T_0; this temperature will fix the coefficients μ_0 and κ_0. After a sufficient interval of time has elapsed since the start of the motion, steady velocity and temperature distributions will be established at the entrance to the pipe ($x = 0$). Depending on the distance from the leading edge, the nature of the velocity, temperature and pressure distributions across the section will change as a result of the no-slip condition together with thermal effects at the walls of the pipe.

If we consider a station in the pipe close to the entrance, and if we assume that the relative pressure difference $(p_0 - p_2)/p_0$ is small, then the effect of the terms discarded in obtaining equations (1.9) from the corresponding boundary-layer equations may be considered small. With these assumptions there is some justification for applying the approximate equations (2.1) to the flow of a gas in a cylindrical pipe in the example just considered.

If we consider the flow of a gas in a pipe laid in the ground, then it is necessary in addition to allow for thermal effects of the earth on the walls of the pipe. If we denote the temperature of the pipe at the entrance by T_1 and the temperature of the ground by T_2, where it is assumed that $T_1 > T_2$, then we may suppose as a first approximation that the temperature of the pipe wall varies according to an exponential law (analogous to the law of Shukhov for oil pipe lines); that is,

$$T_{\text{ст}} = T_2 + (T_1 - T_2)\exp\left(-\beta\frac{x}{a}\right) \tag{2.2}$$

where a is the radius of the pipe and β is a dimensionless coefficient to be determined experimentally.

For simplicity we will assume that the velocity and temperature distributions are uniform at the entrance, at which point the boundary conditions will have the form

$$u = U_1, \quad T = T_1, \quad p = p_1 \quad \text{for } x = 0 \text{ and } 0 < r < a \tag{2.3}$$

The no-slip condition and the axial symmetry of the flow may be exhibited by means of the following equalities;

$$\begin{aligned} &u = 0, \quad v_r = 0 && \text{for } r = a \text{ and } x > 0 \\ &v_r = 0, \quad \frac{\partial u}{\partial r} = 0, \quad \frac{\partial T}{\partial r} = 0 && \text{for } r = 0 \text{ and } x > 0 \end{aligned} \tag{2.4}$$

If we change to dimensionless variables and put

$$\begin{gathered} x = ax_1, \quad r = ar_1, \quad u = U_0u_1, \quad v = U_0v_1, \quad T = T_0\overline{T}, \quad p = p_0p \\ \frac{\rho_0 a U_0}{\mu_0} = R, \quad \frac{\rho_0 U_0^2}{\gamma p_0} = M^2, \quad \frac{AU_0^2}{gC_pT_0} = (\gamma - 1)M^2, \quad \frac{gC_p\mu_0}{\varkappa_0} = P \end{gathered} \tag{2.5}$$

then the differential equations (2.1) and boundary conditions (2.3), (2.2) and (2.4) take the following form:

$$R\left(\frac{\partial u_1}{\partial x_1}+\frac{1}{\gamma M^2}\frac{\partial \bar{p}}{\partial x_1}\right)=\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial u_1}{\partial r_1}\right),\qquad \frac{\partial \bar{p}}{\partial r_1}=0 \tag{2.6}$$

$$PR\left[\frac{\partial \bar{T}}{\partial x_1}-\frac{\gamma-1}{\gamma}\frac{\partial \bar{p}}{\partial x_1}\right]=\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial \bar{T}}{\partial r_1}\right)$$

$$\frac{\partial}{\partial x_1}(u_1+\bar{p}-\bar{T})+\frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1 v_1)=0$$

$$u_1=\frac{U_1}{U_0},\quad \bar{p}=\frac{p_1}{p_0},\quad \bar{T}=\frac{T_1}{T_0},\qquad \text{for } x_1=0 \text{ and } 0<r_1<1$$

$$u_1=0,\quad v_1=0,\quad \bar{T}=\frac{T_2}{T_0}+\frac{T_1-T_2}{T_0}e^{-\beta x_1}\ \text{for } r=a \text{ and } x_1>0 \tag{2.7}$$

$$v_1=0,\quad \frac{\partial u}{\partial r_1}=0,\quad \frac{\partial \bar{T}}{\partial r_1}=0\qquad \text{for } r=0 \text{ and } x_1>0$$

We will solve the system of linear equations (2.6) with the boundary conditions (2.7) with the aid of the method of operational calculus. We put

$$\int_0^\infty e^{-\lambda x_1}u_1 dx_1=\frac{u^*}{\lambda},\quad \int_0^\infty e^{-\lambda x_1}v_1 dx_1=\frac{v^*}{\lambda},\quad \int_0^\infty e^{-\lambda x_1}\bar{p}\,dx_1=\frac{p^*}{\lambda},\quad \int_0^\infty e^{-\lambda x_1}\bar{T}\,dx_1=\frac{T^*}{\lambda} \tag{2.8}$$

Considering the first of the boundary conditions (2.7), we will have

$$\int_0^\infty e^{-\lambda x_1}\frac{\partial u_1}{\partial x_1}dx_1=u^*-\frac{U_1}{U_0},\qquad \int_0^\infty e^{-\lambda x_1}\frac{\partial \bar{p}}{\partial x_1}dx_1=p^*-\frac{p_1}{p_0}$$

$$\int_0^\infty e^{-\lambda x_1}\frac{\partial \bar{T}}{\partial x_1}dx_1=T^*-\frac{T_1}{T_0} \tag{2.9}$$

$$\int_0^\infty e^{-\lambda x_1}[T_2+(T_1-T_2)e^{-\beta x_1}]\,dx_1=\frac{T_2}{\lambda}+\frac{T_1-T_2}{\lambda+\beta}$$

When we construct the Laplace transformation of the equations (2.6) and the boundary conditions (2.7), taking into account (2.8) and (2.9), we obtain the following transformed equations and boundary conditions;

$$R\lambda\left[u^*-\frac{U_1}{U_0}+\frac{1}{\gamma M^2}\left(p^*-\frac{p_1}{p_0}\right)\right]=\frac{d^2u^*}{dr_1^2}+\frac{1}{r_1}\frac{du^*}{dr_1}$$

$$PR\lambda\left[T^*-\frac{T_1}{T_0}-\frac{\gamma-1}{\gamma}\left(p^*-\frac{p_1}{p_0}\right)\right]=\frac{d^2T^*}{dr_1^2}+\frac{1}{r_1}\frac{dT^*}{dr_1}$$

$$\lambda\left(u^*+p^*-T^*-\frac{U_1}{U_0}-\frac{p_1}{p_0}+\frac{T_1}{T_0}\right)=-\frac{1}{r_1}\frac{d}{dr_1}(r_1, v^*) \tag{2.10}$$

$$v^*=0,\quad \frac{\partial u^*}{\partial r_1}=0,\quad \frac{dT^*}{dr_1}=0\qquad \text{for } r_1=0$$

$$u^*=0,\quad v^*=0,\quad T^*=\frac{1}{T_0}\left[T_2+\frac{\lambda}{\lambda+\beta}(T_1-T_2)\right]\qquad \text{for } r_1=1 \tag{2.11}$$

The solution of equation (2.10) corresponding to the first boundary condition (2.11) will have the form

$$u^* = C_1 I_0 (r_1 \sqrt{\lambda R}) + \frac{U_1}{U_0} - \frac{1}{\gamma M^2}\left(p^* - \frac{p_1}{p_0}\right)$$

$$T^* = C_2 I_0 (r_1 \sqrt{\lambda P R}) + \frac{T_1}{T_0} + \frac{\gamma - 1}{\gamma}\left(p^* - \frac{p_1}{p_0}\right) \qquad (2.12)$$

$$v^* \frac{r_1}{\lambda} = \frac{1}{2} r_1^2 \left(\frac{U_1}{U} + \frac{p_1}{p_0} - \frac{T_1}{T_0} - p^*\right) - \int_0^{r_1} (u^* + T^*)\, r_1\, dr_1$$

Using the second boundary condition (2.11), we obtain the following expressions for the transforms of the velocity components and of the temperature;

$$u^* = \left[\frac{U_1}{U_0} - \frac{1}{\gamma M^2}\left(p^* - \frac{p_1}{p_0}\right)\right]\left[1 - \frac{I_0 (r_1 \sqrt{\lambda R})}{I_0 (\sqrt{\lambda R})}\right) \qquad (2.13)$$

$$T^* = \frac{T_1}{T_0} + \frac{\gamma - 1}{\gamma}\left(p^* - \frac{p_1}{p_0}\right) - \left[\frac{\beta}{\lambda + \beta}\, \frac{T_1 - T_2}{T_0} + \frac{\gamma - 1}{\gamma}\left(p^* - \frac{p_1}{p_0}\right)\right] \frac{I_0 (r_1 \sqrt{\lambda P R})}{I_0 (\sqrt{\lambda P R})}$$

The recurrence formula

$$\int_0^x I_0 (x)\, x dx = x I_1 (x) \qquad (2.14)$$

is well known in the theory of Bessel functions. Evaluating the integrals (2.12) with the aid of (2.13) and (2.14) and using the boundary condition (2.11) for the lateral velocity component, we obtain the following equation for the transform of the pressure:

$$p^* - \frac{p_1}{p_0} = \gamma \frac{\dfrac{2U_1}{U_0 \sqrt{\lambda R}}\, \dfrac{I_1 (\sqrt{\lambda R})}{I_0 (\sqrt{\lambda R})} + \dfrac{2\beta}{\lambda + \beta}\, \dfrac{T_2 - T_1}{T_0}\, \dfrac{1}{\sqrt{\lambda P R}}\, \dfrac{I_1 (\sqrt{\lambda P R})}{I_0 (\sqrt{\lambda P R})}}{1 + \dfrac{2(\gamma - 1)}{\sqrt{\lambda P R}}\, \dfrac{I_1 (\sqrt{\lambda P R})}{I_0 (\sqrt{\lambda P R})} - \dfrac{1}{M^2}\left[1 - \dfrac{2}{\sqrt{\lambda R}}\, \dfrac{I_1 (\sqrt{\lambda R})}{I_0 (\sqrt{\lambda R})}\right]} \qquad (2.15)$$

Thus the complete solution of the transformed problem is contained in equations (2.13) and (2.15). We may invert the transformation to obtain the original functions, generally speaking, by the method of decomposition into simple fractions, provided that we can somehow find the roots of the denominator of (2.15).

Inasmuch as the original equations (2.1) only describe the flow of the gas in the entrance region with a certain degree of approximation; that is, for not too large values of the coordinate x_1, and inasmuch as small values of the factor x_1 in the exponent of the transformation formula (2.8) correspond to large values of the transformation parameter λ, then with a certain error we may substitute for the Bessel function of imaginary argument in (2.15) the first term of an asymptotic expansion. That is, we may put

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \qquad \frac{I_1(\sqrt{\lambda R})}{I_0(\sqrt{\lambda R})} \approx 1, \qquad \frac{I_1(\sqrt{\lambda P R})}{I_0(\sqrt{\lambda P R})} \approx 1 \tag{2.16}$$

Substituting (2.16) in (2.17), we obtain the following approximate expression for the transform of the pressure;

$$p^* - \frac{p_1}{p_0} = 2\gamma \frac{\dfrac{U_1}{U_0} + \dfrac{T_2 - T_1}{\sqrt{P} T_0} \dfrac{\beta}{\lambda + \beta}}{\sqrt{\lambda R}\left(1 - \dfrac{1}{M^2}\right) + 2\left(\dfrac{\gamma - 1}{\sqrt{P}} + \dfrac{1}{M^2}\right)} \tag{2.17}$$

Expanding the right-hand side of (2.17) in fractions, we obtain

$$\frac{p^*}{\lambda} = \frac{p_1}{p_0}\frac{1}{\lambda} + \frac{C_1}{\alpha^2}\left(\frac{\alpha}{\lambda} - \frac{1}{\sqrt{\lambda}} + \frac{1}{\alpha + \sqrt{\lambda}}\right) + \frac{C_2}{\alpha^2}\left[\frac{}{\lambda} - \frac{1}{\sqrt{\lambda}} + \right.$$
$$\left. + \frac{\beta}{\beta + \alpha^2}\frac{1}{\sqrt{\lambda} + \alpha} - \frac{\alpha^3}{\beta + \alpha^2}\frac{1}{\lambda + \beta} + \frac{\alpha^2}{2(\beta + \alpha^2)}\left(\frac{1}{\sqrt{\lambda} + \sqrt{\beta}} + \frac{1}{\sqrt{\lambda} - \sqrt{\beta}}\right)\right] \tag{2.18}$$

where

$$\alpha = -\frac{2}{\sqrt{PR}}\frac{(\gamma - 1)M^2 + \sqrt{P}}{1 - M^2}$$
$$C_1 = -\frac{2\gamma U_1 M^2}{U_0 \sqrt{R}}\frac{1}{1 - M^2}, \qquad C_2 = \frac{2\gamma M^2}{T_0 \sqrt{PR}}\frac{T_1 - T_2}{1 - M^2} \tag{2.19}$$

Passing from the transform (2.18) to the original function, using the formulas (2.5) to change from dimensionless to dimensional variables, and introducing the usual notation

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du, \qquad \operatorname{erfc}(x) = 1 - \operatorname{erf}(x) \tag{2.20}$$

we obtain the following approximate formula for the variation of the pressure in a gas in the entrance section of a circular cylindrical pipe;

$$p = p_1 + \frac{C_1 p_0}{\alpha}\left[1 - \exp\frac{\alpha^2 x}{a}\operatorname{erfc}\left(\alpha\sqrt{\frac{x}{a}}\right)\right] + \frac{C_2 p_0}{\alpha}\left[1 - \frac{\alpha^2}{\beta + \alpha^2}\exp\left(-\frac{\beta x}{a}\right) - \right.$$
$$\left. - \frac{\beta}{\beta + \alpha^2}\operatorname{erfc}\left(\alpha\sqrt{\frac{x}{a}} + \frac{\alpha\sqrt{\beta}}{\beta + \alpha^2}\exp\frac{\beta x}{a}\operatorname{erf}\left(\sqrt{\beta\frac{x}{a}}\right)\right)\right] \tag{2.21}$$

As an immediate check we may satisfy ourselves that equation (2.21) satisfies the condition (2.3) for the pressure at the pipe entrance.

If the gas flow ahead of the pipe entrance is subsonic; that is, if the inequality

$$1 - M^2 > 0 \tag{2.22}$$

is satisfied, then the quantities a and C_1 defined by equations (2.19) will be negative, and at the same time we will have

$$\operatorname{erfc}\left(\alpha \sqrt{\frac{x}{a}}\right) = 1 + \operatorname{erf}\left(|\alpha| \sqrt{\frac{x}{a}}\right) > 1$$

In this case the second term of the right-hand side of (2.21), representing the influence of viscosity on the pressure variation in the pipe, will always be negative. Thus if the entering flow is subsonic the pressure in the entrance section of a circular cylindrical pipe will always be reduced by the action of viscosity. This situation is well known in gas dynamics ([7], p. 132).

In order to determine the net sign of the third term in (2.21), representing the effect of cooling by heat transfer at the walls on the pressure variation in the entrance region of a cylindrical pipe, we will resort to an expansion of the entire expression in square brackets as a series in powers of the argument. If we stop with the second terms in the expansions for the individual factors, we obtain

$$1 - \frac{1}{\beta + \alpha^2}\left[\alpha^2 \exp\frac{-\beta x}{a} - \beta \exp\frac{\alpha^2 x}{a} \operatorname{erfc}\left(\alpha \sqrt{\frac{x}{a}}\right) + \right.$$
$$\left. + \alpha\sqrt{\beta} \exp\frac{\beta x}{a} \operatorname{erf}\left(\sqrt{\beta\frac{x}{a}}\right)\right] \approx \frac{4\beta\alpha}{\sqrt{\pi}(\beta + \alpha^2)} \sqrt{\frac{x}{a}}\left[1 + \frac{1}{3}(\alpha^2 + \beta)\frac{x}{a}\right]$$

The net sign of the third term in (2.21) depends on the sign of the factor C_2 and of the parameter β. If the inequality (2.22) is satisfied, and if it is assumed that the walls are cooled ($T_2 < T_1$, $\beta > 0$), then the net sign of the third term will be positive. Consequently, if the flow into the entrance of a cylindrical pipe is subsonic, heat removal from the gas by cooling at the wall will lead to an increase in pressure in the entrance region of the pipe. This situation also appears to be generally known in gas dynamics ([8], p. 53).

Thus the qualitative conclusions from formula (2.21) agree with the familiar formulas of gas dynamics. Besides these qualitative conclusions, however, equation (2.21) makes it possible to compute the pressure variation as a function of the distance from the pipe entrance, a computation which is not possible using the formulas of gas dynamics.

In deriving the approximate formula (2.21) for the pressure we used the leading term of the asymptotic formulas (2.16). As can be seen from these formulas, the Reynolds number appears in the argument of the Bessel function. Consequently, the larger the Reynolds number the more accurately will the leading term of the asymptotic formulas represent the value of

Bessel functions, and the greater will be the length of pipe for which the approximate formula (2.21) will be accurate for the pressure. In such cases we may use this formula to estimate the length l of the entrance region of the pipe, or the length required for the pressure to change from the given value p_1 at the entrance to the given value p_2 at the end;

$$\alpha \frac{p_2 - p_1}{p_0} = C_1\left[1 - \exp\frac{\alpha^2 l}{a}\operatorname{erfc}\left(\alpha\sqrt{\frac{l}{a}}\right)\right] + C_2\left\{1 - \frac{\alpha^2}{\beta+\alpha^2}\left[\exp\frac{-\beta l}{\alpha} - \right.\right.$$
$$\left.\left. - \beta\exp\frac{\alpha^2 l}{a}\operatorname{erfc}\left(a\sqrt{\frac{l}{a}}\right) + \alpha\sqrt{\beta}\exp\frac{\beta l}{a}\operatorname{erf}\left(\sqrt{\beta\frac{l}{a}}\right)\right]\right\} \qquad (2.23)$$

Provided that the length l of the entrance section is not large compared to the radius a of the pipe, we may expand the various functions entering into the right-hand side of (2.23) in powers of their arguments. In particular, if we terminate this expansion of the right-hand side of (2.23) with terms containing l/a to the first power, we obtain the following approximate formula for the length of the entrance section;

$$\frac{l}{a} \approx \frac{\pi}{4} - \left[C_1 + \frac{2\beta}{\beta+\alpha^2}C_2\right]^{-2}\left(\frac{p_2-p_1}{p_0}\right)^2 \qquad (2.24)$$

We will now return to the formulas (2.13) for the transforms of the velocity and temperature. Inasmuch as the inverse transform for the pressure has already been found in approximate form, it is sufficient for the determination of the inverse transforms for the velocity and temperature according to (2.15) to find the inverse transforms of the ratios

$$\frac{I_0(r_1\sqrt{\lambda R})}{I_0(\sqrt{\lambda R})}, \qquad \frac{I_0(r_1\sqrt{\lambda PR})}{I_0(\sqrt{\lambda PR})}$$

and then to employ the convolution theorem of the operational calculus ([9], p. 13). For an approximate determination of the inverse transforms, either expansions in powers of the argument or the asymptotic formulas (2.16) may be used.

In the neighborhood of the pipe wall we may put

$$\frac{I_0(r_1\sqrt{\lambda R})}{I_0(\sqrt{\lambda R})} \approx \frac{1}{\sqrt{r_1}}\exp[-\sqrt{\lambda R}(1-r_1)]$$

Then we obtain from equations (2.13)

$$u^* = \frac{U_1}{U_0} - \frac{1}{\gamma M^2}\left(p^* - \frac{p_1}{p_0}\right) - \frac{U_1}{U_0\sqrt{r_1}}\exp[-(1-r_1)\sqrt{\lambda R}] +$$
$$+ \frac{1}{\gamma M^2\sqrt{r}}\left(p^* - \frac{p_1}{p_0}\right)\exp[-(1-r_1)\sqrt{\lambda R}]$$

$$T^* = \frac{T_1}{T_0} + \frac{\gamma-1}{\gamma}\left(p^* - \frac{p_1}{p_0}\right) + \frac{T_2-T_1}{T_0\sqrt{r_1}}\,\frac{\beta}{\lambda+\beta}\exp[-(1-r_1)\sqrt{\lambda PR}] -$$
$$- \frac{\gamma-1}{\gamma\sqrt{r_1}}\left(p^* - \frac{p_1}{p_0}\right)\exp[-(1-r_1)\sqrt{\lambda PR}]$$

The image function $\exp(-a\sqrt{\lambda})$ is the transform of $\operatorname{erfc}(a/2\sqrt{x})$ ([9], p. 152). Using the convolution theorem, we obtain the following approximate expressions for the velocity and temperature in the boundary layer near the walls of the pipe in the entrance region;

$$u = U_1 - \frac{p-p_1}{\gamma p_0 M^2} U_0 - U_1 \sqrt{\frac{a}{r}} \operatorname{erfc}\left[\left(1-\frac{r}{a}\right)\frac{\sqrt{aR}}{2\sqrt{x}}\right] + $$
$$+ \frac{U_0\sqrt{a}}{\gamma M^2 \sqrt{r}} \frac{d}{dx_1}\int_0^{x_1} \operatorname{erfc}\left[\frac{(1-r_1)\sqrt{R}}{2\sqrt{x_1-\xi}}\right]\frac{p-p_1}{p}\,d\xi \qquad (2.25)$$
$$T = T_1 + T_0\,\frac{\gamma-1}{\gamma}\frac{p-p_1}{p_0} + (T_2 - T_1)\sqrt{\frac{a}{r}}\times$$
$$\times\frac{d}{dx_1}\int_0^{x_1}[1-e^{-\beta(x_1-\xi)}]\operatorname{erfc}\left[\left(1-\frac{r}{a}\right)\frac{\sqrt{PR}}{2\sqrt{\xi}}\right]d\xi -$$
$$-\frac{\gamma-1}{\gamma}\sqrt{\frac{a}{r}}\frac{d}{dx_1}\int_0^{x_1}\operatorname{erfc}\left[\left(1-\frac{r}{a}\right)\frac{\sqrt{PR}}{2\sqrt{x_1-\xi}}\right]\frac{p-p_1}{p_0}\,d\xi$$

For the region near the pipe axis ($r_1 = 0$) we may use (2.16) as denominator in (2.13) and a power series expansion as numerator, stopping after the second term. That is, we put

$$I_0(x) \approx 1 + \tfrac{1}{4}x^2$$

Then we obtain from (2.13)

$$u^* = \frac{U_1}{U_0} - \frac{1}{\gamma M^2}\left(p^* - \frac{p_1}{p_0}\right) - \frac{U_1}{U_0}\sqrt{2\pi R^{1/2}}\left[\lambda^{1/4} + \tfrac{1}{4}r_1^2\lambda^{5/4}\right]e^{-\sqrt{\lambda R}} +$$
$$+\frac{\sqrt{2\pi R^{1/2}}}{\gamma M^2}\left(p^* - \frac{p_1}{p_0}\right)\left(\lambda^{1/4} + \tfrac{1}{4}r_1^2\lambda^{5/4}\right)e^{-\sqrt{\lambda R}} \qquad (2.26)$$
$$T^* = \frac{T_1}{T_0} + \frac{\gamma-1}{\gamma}\left(p^* - \frac{p_1}{p_0}\right) - \frac{T_2 - T_1}{T_0}\sqrt{2\pi(PR)^{1/2}}\frac{\beta}{\lambda+\beta}\times$$
$$\times\left(\lambda^{1/4} \quad \tfrac{1}{4}r_1^2\lambda^{5/4}\right)e^{-\sqrt{\lambda PR}} +$$
$$+\frac{\gamma-1}{\gamma}\sqrt{2\pi(PR)^{1/2}}\left(p^* - \frac{p_1}{p_0}\right)\left(\lambda^{1/4} + \tfrac{1}{4}r_1^2\lambda^{5/4}\right)e^{-\sqrt{\lambda PR}}$$

To find the inverse transformation of (2.26) we may use formula (3.91) from the book ([9], p. 154), according to which the transform

$$\lambda^{\nu/2}e^{-\alpha\sqrt{\lambda}}$$

corresponds to the original function

$$\sqrt{\frac{2}{\pi}}(2x_1)^{-1/2\nu}\exp\frac{-\alpha^2}{8x_1}D_{\nu-1}\left(\frac{\alpha}{\sqrt{2x_1}}\right)$$

where D_n is the Weber function. In the case we are considering the arguments of the Weber functions will be

$$\sqrt{\frac{R}{2x_1}}, \qquad \sqrt{\frac{kP}{2x_1}}$$

As was remarked earlier, for large Reynolds numbers R and for small distances x_1 from the entrance the arguments of the Weber functions will be sufficiently large so that we may use the leading term of the asymptotic expansion ([10], p. 347);

$$D_n(z) \sim z^n \exp\left(-\frac{z^2}{4}\right)$$

Thus the transforms $\lambda^{1/4} \exp[-\sqrt{(\lambda R)}]$ and $\lambda^{5/4} \exp[-\sqrt{(\lambda R)}]$ will correspond approximately to the following functions;

$$\sqrt{\frac{2}{\pi}} R^{-1/4} \exp\frac{-R}{4x_1}, \qquad \sqrt{\frac{2}{\pi}} (2x_1)^{-3/2} R^{3/4} \exp\frac{-R}{4x_1} \tag{2.27}$$

For small values of x_1 and large values of R the reduced expressions (2.27) for the original functions will be relatively small, and therefore as a first rough approximation we may neglect all terms in the expressions (2.26) containing the factor $\exp[-\sqrt{(\lambda R)}]$. Proceeding now to the inverse transformation, we obtain

$$u = U_1 - \frac{U_0}{\gamma M^2}\,\frac{p - p_1}{p_0}, \qquad T = T_1 + T_0\,\frac{\gamma - 1}{\gamma}\,\frac{p - p_1}{p_0} \tag{2.28}$$

It follows from the approximate formulas (2.28) that the velocity increases, but the temperature decreases, when the pressure decreases along the axis of the pipe.

In order to obtain more accurate formulas for the variation of the pressure, velocity, and temperature along the pipe we have to expand the right-hand sides of (2.15) and (2.13) as simple fractions according to the roots of the denominator of (2.15). At present it can be said that the nature of these roots is fully established only for the case of Prandtl number P equal to unity. If we use the recurrence formula

$$xI_0(x) = I_1(x) + xI_1'(x)$$

then we obtain in the case $P = 1$ the following transcendental equation from which to determine the roots of the denominator of (2.15);

$$I_1(\sqrt{\lambda R}) + \frac{M^2 - 1}{2\gamma M^2 - M^2 + 1}\sqrt{\lambda R}\, I_1'(\sqrt{\lambda R}) = 0 \tag{2.29}$$

Putting $\sqrt{(\lambda R)} = ix$, we obtain from (2.29)

$$\frac{2\gamma M^2 + 1 - M^2}{M^2 - 1} J_1(x) + xJ_1'(x) = 0 \tag{2.30}$$

Regarding equation (2.30) it is known ([11] , p. 482) that if the inequality

$$\alpha\sqrt{R}+2=\frac{2\gamma M^2+1-M^2}{M^2-1}+1=\frac{2\gamma M^2}{M^2-1}<0 \qquad (2.31)$$

is satisfied then two roots of equation (2.30) will be purely imaginary, and all the remaining roots will be real. The inequality (2.31) will be satisfied on satisfying the inequality (2.22). Consequently if the gas flow ahead of the pipe entrance is subsonic, and if the Prandtl number is equal to unity, then one root of equation (2.29) (with respect to λ) will be real and positive, while an infinite number of roots will be real, negative, and simple. For this reason we obtain, on expanding the right-hand side of (2.15) in simple fractions,

$$\frac{1}{\lambda}\left(p^*-\frac{p_1}{p_0}\right)=\frac{F_1(\lambda)}{F_2(\lambda)}=\frac{[C_1+\beta C_2/(\lambda+\beta)]\,I_1(\sqrt{\lambda R})}{\lambda\,[\sqrt{\lambda}I_0(\sqrt{\lambda R})+\alpha I_1(\sqrt{\lambda R})]}=$$

$$=\frac{A_0}{\lambda}+\frac{A''}{\lambda+\beta}+\frac{A'}{\lambda-\lambda_0}+\sum_{k=\lambda}^{\infty}\frac{A_k}{\lambda-\lambda_k} \qquad (2.32)$$

where

$$A_0=\left(\frac{\lambda F_1}{F_2}\right)_{\lambda=0}=\frac{\sqrt{R}\,(C_1+C_2)}{2+\alpha\sqrt{R}}=\frac{U_1}{U_0}+\frac{T_2-T_1}{T_0}$$

$$A''=-\frac{C_2J_1(\sqrt{\beta R})}{\sqrt{\beta}J_0(\sqrt{\beta R})+\alpha J_1(\sqrt{\beta R})} \qquad (2.33)$$

$$A'=\frac{F_1(\lambda_0)}{F_2'(\lambda_0)}=\frac{2\,[C_1+\beta C_2/(\beta+\lambda)]}{\lambda_0\sqrt{R}-2\alpha-\alpha^2\sqrt{R}}, \qquad A_k=\frac{F_1(\lambda_k)}{F_2'(\lambda_k)}=\frac{2\,[C_1+\beta C_2/(\beta+\lambda_k)]}{\lambda_k\sqrt{R}-2\alpha-\alpha^2\sqrt{R}}$$

Substituting the expression (2.15) in (2.13), we obtain as transforms for the velocity and temperature

$$\frac{u^*}{\lambda}=\frac{\varphi_1(\lambda)}{\varphi_2(\lambda)}=\frac{U_1}{U_0}\left[\frac{1}{\lambda}-\frac{I_0(r_1\sqrt{\lambda R})}{\lambda J_0(\sqrt{\lambda R})}\right]-$$

$$-\frac{1}{\gamma M^2}\left[\frac{1}{\lambda}-\frac{I_0(r\sqrt{\lambda R})}{\lambda I_0(\sqrt{\lambda R})}\right]\frac{[C_1+\beta C_2/(\beta+\lambda)]\,I_1(\sqrt{\lambda R})}{\sqrt{\lambda}I_0(\sqrt{\lambda R})+\alpha I_1(\lambda R)}=$$

$$=\frac{B_0}{\lambda}+\frac{B''}{\lambda+\beta}+\frac{B'}{\lambda-\lambda_0}+\sum_{k=1}^{\infty}\frac{B_k}{\lambda-\lambda_k}+\sum_{k=1}^{\infty}\frac{B_k'}{\lambda-\lambda_k'} \qquad (2.34)$$

$$\frac{T}{\lambda}=\frac{\psi_1(\lambda)}{\psi_2(\lambda)}=\frac{T_1}{T_0}\,\frac{1}{\lambda}-\frac{\beta\,(T_1-T_2)}{\lambda(\lambda+\beta)\,T_0}\,\frac{I_0(r_1\sqrt{\lambda R})}{I_0(\sqrt{\lambda R})}+$$

$$+\frac{\gamma-1}{\gamma\lambda}\left[1-\frac{I_0(r_1\sqrt{\lambda R})}{I_0(\sqrt{\lambda R})}\right]\frac{[C_1+\beta C_2/(\beta+\lambda)]\,I_1(\sqrt{\lambda R})}{\sqrt{\lambda}I_0(\sqrt{\lambda R})+\alpha I_1(\sqrt{\lambda R})}=$$

$$=\frac{D_0''}{\lambda}+\frac{D''}{\lambda+\beta}+\frac{D'}{\lambda-\lambda_0}+\sum_{k=1}^{\infty}\left(\frac{D_k}{\lambda-\lambda_k}+\frac{D_k'}{\lambda-\lambda_k'}\right)$$

where the λ_k are the negative roots of equation (2.29) and the λ_k' are connected with the roots of the equation $J_0(\nu) = 0$ by the relationship

$$\lambda_k' = -\frac{\nu_k^2}{R} \qquad (2.35)$$

The coefficients entering in the expansion (2.34) are of the form

$$B_0 = \left[\frac{\lambda\varphi_1(\lambda)}{\varphi_2(\lambda)}\right]_{\lambda=0} = 0, \qquad B'' = -\frac{A'}{\gamma M^2}\left[1 - \frac{J_0(r_1\sqrt{\beta R})}{J_0(\sqrt{\beta R})}\right]$$

$$B' = -\frac{A'}{\gamma M^2}\left[1 - \frac{I_0(r_1\sqrt{\lambda_0 R})}{I_0(\sqrt{\lambda_0 R})}\right], \qquad B_k = -\frac{A_k}{\gamma M^2}\left[1 - \frac{I_0(r_1\sqrt{\lambda_k R})}{I_0(\sqrt{\lambda_k R})}\right]$$

$$B_k' = \frac{2U_1}{U_0\nu_k}\frac{J_0(r_1\nu_k)}{J_1(\nu_k)} - \frac{2}{\gamma a M^2\nu_k}\left(C_1 + \frac{\beta}{\beta - (\nu_k^2/R)}C_2\right)\frac{J_0(r_1\nu_k)}{J_1(\nu_k)} \qquad (2.36)$$

$$D_0 = \left[\frac{\lambda\psi_1(\lambda)}{\psi_2(\lambda)}\right]_{\lambda=0} = \frac{T_2}{T_0}, \qquad D' = \frac{\gamma-1}{\gamma}A'\left[1 - \frac{I_0(r_1\sqrt{\lambda_0 R})}{I_0(\sqrt{\lambda_0 R})}\right.$$

$$D'' = \frac{T_1 - T_2}{T_0}\frac{J_0(r_1\sqrt{\beta R})}{J_0(\sqrt{\beta R})} + \frac{\gamma+1}{\gamma}A''\left[1 - \frac{J_0(r_1\sqrt{\beta R})}{J_0(\sqrt{\beta R})}\right]$$

$$D_k = \frac{\gamma-1}{\gamma}A_k\left[1 - \frac{I_0(r_1\sqrt{\lambda_k R})}{I_0(\sqrt{\lambda_k R})}\right]$$

$$D_k' = \frac{2\beta(T_1 - T_2)}{T_0(\beta - \nu_k^2/R)\nu_k}\frac{J_0(r_1\nu_k)}{J_1(\nu_k)} + \frac{2(\gamma-1)}{a\gamma\nu_k}\left(C_1 + \frac{\beta}{\beta - (\nu_k^2/R)}C_2\right)\frac{J_0(r_1\nu_k)}{J_1(\nu_k)}$$

If we denote the real roots of equation (2.30) by γ_k; that is, if we put

$$\sqrt{\lambda_k R} = i\gamma_k, \quad \lambda_k = -\frac{\gamma_k^2}{R} \qquad (2.37)$$

and if we proceed from the transforms (2.32) and (2.34) to the original functions, we obtain the following formulas for pressure, velocity and temperature:

$$\frac{p}{p_0} = \frac{p_1}{p_0} + \frac{U_1}{U_0} - \frac{T_1 - T_2}{T_0} + A'\exp\frac{x\lambda_0}{a} + A''\exp\frac{-\beta x}{a} + \sum_{k=1}^{\infty} A_k\exp\left(-\frac{x}{a}\frac{\gamma_k^2}{R}\right)$$

$$\frac{U}{U_0} = -\frac{A'}{\gamma M^2}\left[1 - \frac{I_0[(r/a)\sqrt{\lambda_0 R}]}{I_0(\sqrt{\lambda R})}\right]\exp\frac{x\lambda_0}{a} + B^{11}\exp\frac{-\beta x}{a} +$$
$$+ \sum_{k=1}^{\infty}\left[B_k\exp\left(-\frac{x}{a}\frac{\gamma_k^2}{R}\right) + B_k'\exp\left(-\frac{x}{a}\frac{\nu_k^2}{R}\right)\right] \qquad (2.38)$$

$$\frac{T}{T_0} = \frac{T_2}{T_0} + \frac{\gamma-1}{\gamma}A'\left[1 - \frac{I_0[(r/a)\sqrt{\lambda_0 R}]}{I_0(\sqrt{\lambda R})}\right]\exp\frac{x\lambda_0}{a} + D''\exp\frac{-\beta x}{a} +$$
$$+ \sum_{k=1}^{\infty}\left[D_k\exp\left(-\frac{x}{a}\frac{\gamma_k^2}{R}\right) + D_k\exp\left(-\frac{x}{a}\frac{\nu_k^2}{R}\right)\right]$$

As was remarked above, the approximate equations (2.1) may be considered correct only for the entrance section of the pipe. For this reason, arbitrarily large values cannot be assigned to the ratio x/a in the formulas (2.38) just obtained. Owing to the presence of the positive real root λ_0, the terms in (2.38) with the factor $\exp(\lambda_0 x/a)$ will grow indefinitely; however, the sign of these terms depends on the sign of the coefficient A'. We will now show that the sign of the denominator of the expression for A' will always be positive. According to the notation of (2.32) we have

$$F_2(x) = x^3\left[I_0(x) - \frac{|\alpha|\sqrt{R}}{x} I_1(x)\right] \tag{2.39}$$

Inasmuch as

$$I_0(0) = 1, \qquad \left[\frac{1}{x} I_1(x)\right]_{x=0} = \frac{1}{2}, \qquad |\alpha|\sqrt{R} > 2$$

then the graph of the expression in brackets in (2.39) will approach the axis of abscissas from below at the point $x = \sqrt{(\lambda_0 R)}$. That is,

$$\left[I_0(x) - \frac{|\alpha|\sqrt{R}}{x} I_1(x)\right]_{x=0} < 0$$

and therefore

$$F_2'(\sqrt{\lambda_0 R}) > 0$$

Thus the sign of the coefficient A' will be determined by the sign of the numerator in (2.33). However, because the inequality (2.22) is satisfied the sign of the numerator will depend on the sign of the following expression:

$$\beta\,\frac{T_1 - T_2}{T_0} - (\beta + \lambda_0)\,\frac{U_1}{U_0} \tag{2.40}$$

The expression (2.40) will certainly be negative if the inequality

$$\frac{U_1}{U_0} > \frac{T_1 - T_2}{T_0} \tag{2.41}$$

is satisfied. Consequently, if the gas flow ahead of the pipe entrance is subsonic and if the inequality (2.41) is satisfied, then the pressure in the pipe can only decrease. At a certain distance from the entrance this pressure will become negative and will increase in magnitude according to an exponential law. Under these conditions the temperature will behave in a similar way, while the velocity of the fluid particles on the axis of the pipe will increase indefinitely.

If we compare this behavior of the pressure and velocity during the development of a gas flow in a pipe with the behavior of these quantities during the development of the flow of an incompressible fluid, we can detect an essential difference. In the solution for the problem of flow of an incompressible fluid in a pipe the pressure always decreases, but the gradient of pressure at an infinite distance from the entrance becomes constant; for this reason the velocity on the axis of the pipe only grows to a limiting value equal to twice the mean velocity. In the solution obtained for the problem of flow of a gas in a pipe, the pressure gradient at an infinite distance from the entrance does not become constant, but the magnitude of this gradient increases indefinitely; hence the velocity on the axis also grows indefinitely. The latter circumstance excludes the possibility that the method used to determine a representative length for the entrance region for the flow of an incompressible fluid can be extended to gas flows.

The representative length for the entrance region for a gas flow in a pipe has to be introduced either through the pressure or through the temperature. If we use the fact that the pressure cannot become negative, for example, we can obtain from the first equation (2.38) with $p = 0$ a formula for the determination of the possible initial length l_* of the pipe. Putting $p = p_2$ and $x = l$ in the first equation (2.38), we obtain a relationship connecting the three undetermined parameters U_1, p_1, and T_1 at the entrance with the given parameters U_0, p_0, and T_0 and the given quantities p_2 and l. For the case in which the difference $(p_0 - p_2)/p_0$ may be considered small we may put $p_1 = p_0$, $T_1 = T_0$. Then the discharge of gas will be determined by the formula

$$Q = \rho_0 \frac{\pi a^2}{2} U_1 \tag{2.42}$$

In other cases we may use the same considerations which have been used in gas dynamics ([7], p. 48).

3. If we write the equations for a stationary two-dimensional gas flow, neglecting viscosity and linearizing the transport terms in the manner of Oseen, we obtain the following approximate equations;

$$U\frac{\partial u}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \qquad U\frac{\partial v}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \qquad \frac{U}{\rho_0 a^2}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.1}$$

These equations have been applied to the flow around a thin profile both for the case $M < 1$ and for the case $M > 1$. Equations (1.9) differ from (3.1) in that certain terms are included to account for viscosity, while the inertial term $U\partial v/\partial x$ is dropped in conformity with the assumptions of the boundary layer. Consequently, equation (1.9) may also be applied to the case $M > 1$. The formulas obtained in Part 1 will be suitable for the case $M > 1$ except for the conclusions which were established by the use of the inequality (2.22). In the case $M > 1$ ahead of the pipe entrance a shock wave will appear, and the quantities U_1, p_1, and T_1 for the gas at the entrance will then be connected with the quantities U_0, p_0, and T_0 by the usual relationships for shock waves.

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